Numerical Differentiation For Adaptively Refined Finite Element Meshes

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Abstract

Postprocessing of point-wise data is a fundamental process in many fields of research. Numerical differentiation is a key operation in computational electromagnetics. In the case of data obtained from a finite element method with automatic mesh refinement much work needs still to be done. This paper addresses some issues in differentiating data obtained from a finite element electromagnetic code with adaptive mesh refinement, and it proposes a methodology for deriving the electric field given the magnetic field on a mesh of linear triangular elements. The procedure itself is nevertheless more general and might be extended for numerically differentiating any point-wise solution based on triangular meshes.

Introduction

Of all the finite element postprocessing operations, numerical differentiation is one of the most important and for this reason it has been under investigation for many years[1-6]. However, numerical differentiation is a notoriously difficult operation prone to error. In order to keep errors under control, much effort has been invested by many researchers. According to Szabo and Babuska[7], all the methods can be separated in three groups: direct differentiation methods, smoothing methods based on superconvergence properties and method based on integral transformations. The choice of the method depends mostly on the application and on the use of the derivatives. Nevertheless, no special attention has been dedicated so far to numerical differentiation of data from adaptively refined meshes in electromagnetic applications.

This paper emphasizes first order differentiation of finite element solutions based on a highly irregular triangular mesh, such as those obtainable from an adaptive mesh refinement process, but it also addresses general numerical issues concerning a wider set of point-wise solutions (such as from experimental data).

Finite element solutions are defined in terms of the local finite element functions, so the simplest natural approach to differentiation is to differentiate the locally valid approximated function. However, the common case of first order finite elements would yield constant derivative on each element and therefore a discontinuous solution over the entire mesh. Various forms of local averaging between elements have been used to estimates derivatives. Methods based on integral transformations (on Green's second identity)[8] are extremely accurate even for third and fourth derivatives, but computationally very expensive.

In general, to obtain good accuracy it is necessary to write a relatively complex postprocessing code and this sometimes makes numerical differentiation unattractive. Nevertheless, derivatives can always be constructed at low cost by differentiating the interpolative polynomial approximation of the point-wise data obtained via the finite element method. This approach often works well for first derivatives, especially for structured meshes or for unstructured meshes with a small degree of anisotropy (i.e. where the density of mesh doesn't change abruptly). However, meshes obtained from adaptive mesh refinement) often show a pronounced irregularity, due to the simultaneous presence of some areas that are finely refined and others that are coarsely refined (see Fig.1).

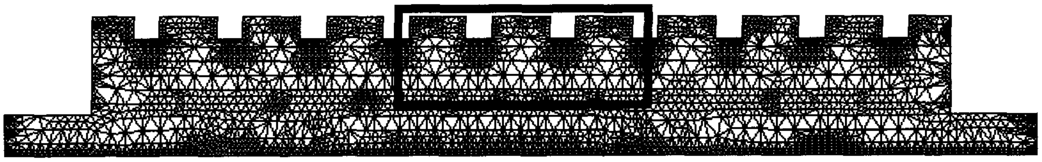
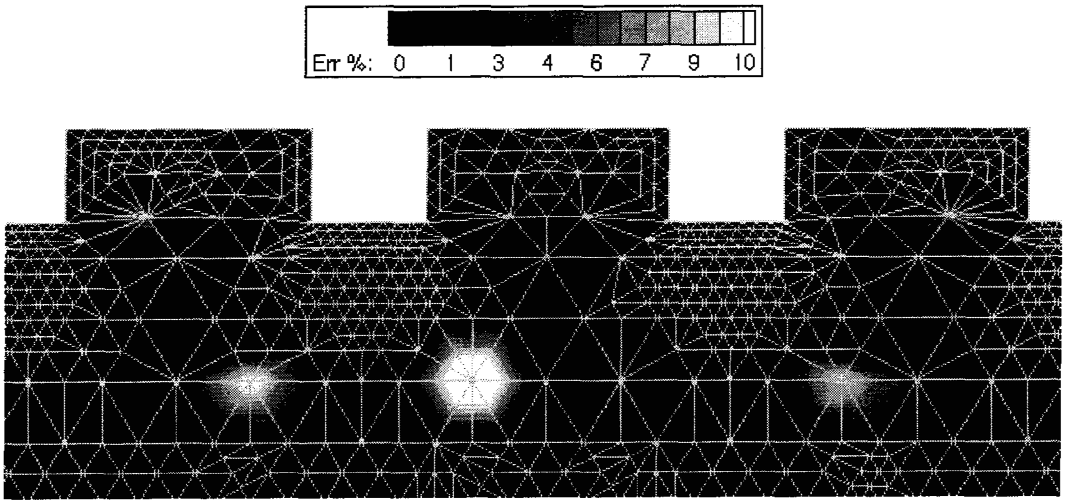


Fig.1 - A typical mesh structure obtained from ad adaptive mesh refinement (detailed view of the area within the box is shown in Fig.2)

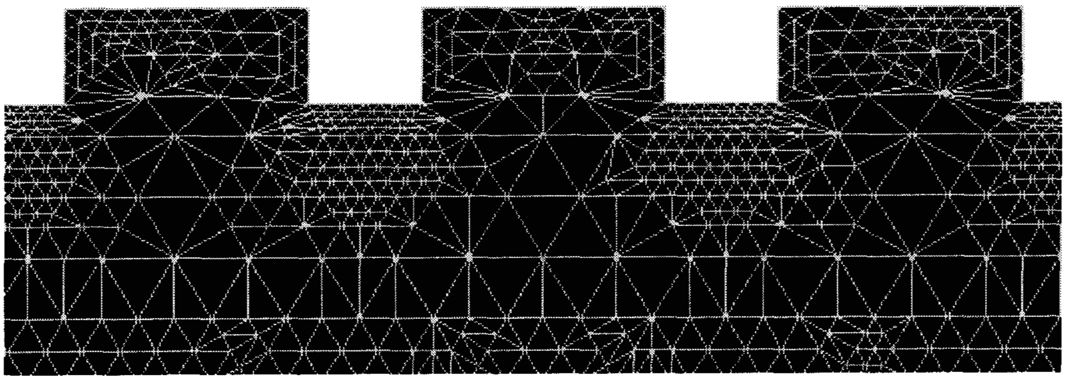
The interpolation of such highly irregular meshes exhibits a certain numerical instability, so that an algorithm that works well for a regular finite element mesh might not work as well for an adaptively refined mesh. The reason of this is purely numerical. Considering triangular meshes, the derivative evaluated at a certain point can be obtained from the polynomial that locally fits the vertices of all the elements that surround that point. Such a local polynomial is of fixed order p and it is computed by solving a linear system with a least-squares algorithm. Since the areas of transition between refined regions and coarse regions are characterized by a large number of relatively distant nodes (Fig.1) , it might happen that a polynomial of order p cannot fit these distant points if they exhibit a wide range in the value of the field. The result is an interpolating polynomial that accurately reconstructs the field locally everywhere except in these particular points, where it might exhibit a relevant error (Fig. 2.a).

Using 'centroidal' values

The key issue in the polynomial interpolation is the choice of the points to be interpolated. The natural way to do this is to choose the vertices of each triangle that share a given point. In this manner it is possible to obtain a set of completely independent data as known terms of the local solving system. Nevertheless a wide range in the values of the field in the vertexes of the triangles might cause numerical instability in the solution of the local system (Fig.2a), so an average of those values is required. A good solution appears to be using the value in the center of each triangle rather than at its vertex. In the case of linear triangular elements this means arithmetically averaging the values at the three vertices of the triangle. This indeed reduce the swing of the values at the chosen points and it dramatically improves the robustness of the methodology (Fig.2b).



a)



b)

Fig. 2 - Detailed view from Fig. 1: difference in percentage between the original field and the interpolated field, in the case of:

a) - interpolation using the vertices of the surrounding elements.

b) - interpolation using the 'centroidal' points of the surroinding elements.

This paper will present results from particular applications. Specific models will be considered and an adaptive mesh refinement procedure will be applied to them in order for the magnetic field to be computed using an existing finite element code. Then the electric field will be computed by interpolating the 'centroidal' values of the triangles and results will be shown to be reliable.

Conclusions

The paper addresses some issues in differentiating data from highly irregular meshes. A smoothing procedure for differentiating data from an adaptive mesh refinement is presented and results are shown to verify its usefulness. The methodology has been successful for a electromagnetic modeling and it appears to be suitable for any problem where an adaptive mesh refinement procedure is used.

Aknowledgment

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